Algebraic Stability of Zigzag Modules.

(By Botnan & Lesnick).

Motivation (Persistent homology vs Zigzag Persistent homology).

Let T be a topological space.

Let $\gamma : T \to \mathbb{R}$ a function. Let $S^{\uparrow}(\gamma)$ be the subfiltration of γ, i.e.

$S^{\uparrow}(\gamma) = \{\gamma^{-1}(-\infty, a] : a \in \mathbb{R}\}$. Note that $\gamma^{-1}(-\infty,a] \subseteq \gamma^{-1}(-\infty,b]$

whenever $a \le b$.

Persistent homology.

(Stability for functions). Ⓘ

Let T be a topological space. Let $\gamma, \kappa : T \to \mathbb{R}$ such that $H_i S^{\uparrow}(\gamma)$ & $H_i S^{\uparrow}(\kappa)$ are p.f.d.

Then we have,

$$d_B(\mathrm{dgm}(H_i S^{\uparrow}(\gamma)), \mathrm{dgm}(H_i S^{\uparrow}(\kappa))) \le \|\gamma - \kappa\|_\infty.$$

Generalize

(Isometry theorem) Ⓘⓘⓘ

For p.f.d persistence modules

$M, N : (\mathbb{R}, \le)$ or $(\mathbb{Z}, \le) \longrightarrow \mathrm{Vec}$,

$$d_B(\mathrm{dgm}(M), \mathrm{dgm}(N)) = d_I(M, N).$$

Remark ① Purely algebraic,

② By this theorem, Ⓘ becomes trivial.

(∵) $\gamma^{-1}(-\infty, a] \hookrightarrow \kappa^{-1}(-\infty, a+\epsilon]$

&

$\kappa^{-1}(-\infty, a] \hookrightarrow \gamma^{-1}(-\infty, a+\epsilon]$ $\forall a \in \mathbb{R}$.

③ Using interleaving distance

We are able to compare two real valued functions defined on different domains.

ZZ persistent homology

Ⓘⓘ

Let $\gamma, \kappa : T \to \mathbb{R}$ be Morse type. Then

$$\underbrace{d_B(\mathrm{dgm}(H_\bullet \mathcal{L}^{\leftrightarrow}(\gamma)), \mathrm{dgm}(H_\bullet \mathcal{L}^{\leftrightarrow}(\kappa)))}_{\hookrightarrow \text{ will be discussed later on.}} \le \|\gamma - \kappa\|_\infty$$

Generalize

Ⓘⓥ

Goal We will prove.

For P.f.d Zigzag modules

$M, N : \mathbb{ZZ} \longrightarrow \mathrm{Vec}$

$$d_B^{ZZ}(\mathrm{dgm}^{ZZ}(M), \mathrm{dgm}^{ZZ}(N)) = d_I^{ZZ}(M, N)$$

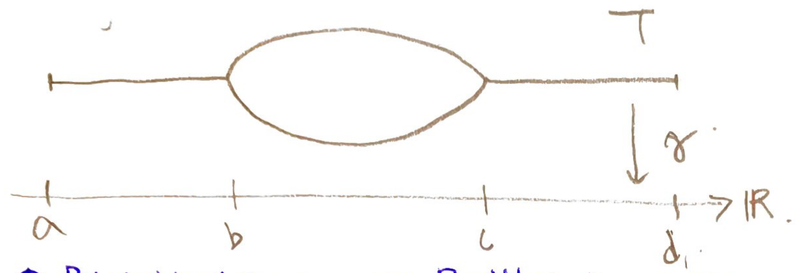
① Purely algebraic

② By this theorem, Ⓘⓘ becomes trivial.

③ //

About Zigzag persistent homology (by example)

Consider $\gamma : T \to \mathbb{R}$ depicted as follows.



① By sublevel filtration & Persistent homology.

For all $r \in [a, \infty)$, $H_0(\gamma^{-1}(-\infty, r); \mathbb{F}) \cong \mathbb{F}$.

Also, observe that $\mathrm{dgm}(H_0(\mathcal{S}^{\uparrow}(\gamma))) = I^{[a,\infty)}$.

② ZZ - Persistent homology.

Consider

$$\begin{tikzcd}[column sep=tiny]
 & \gamma^{-1}(-\infty,b) & & \gamma^{-1}(a,c) & & \gamma^{-1}(b,d) & & \gamma^{-1}(c,\infty) & \\
\gamma^{-1}(-\infty,a) \arrow[ur] & & \gamma^{-1}(a,b) \arrow[ul] \arrow[ur] & & \gamma^{-1}(b,c) \arrow[ul] \arrow[ur] & & \gamma^{-1}(c,d) \arrow[ul] \arrow[ur] & & \gamma^{-1}(d,\infty) \arrow[ul]
\end{tikzcd}$$

$\gamma^{-1}(-\infty,a) = \phi$, $\gamma^{-1}(d,\infty) = \phi$

$\downarrow H_0$

$$\begin{tikzcd}[column sep=small]
 & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & \\
0 \arrow[ur] & & \mathbb{F} \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & & \mathbb{F}^2 \arrow[ul, "(1\ 1)"] \arrow[ur, "(1\ 1)"'] & & \mathbb{F} \arrow[ul, "\cong"] \arrow[ur, "\cong"'] & & 0 \arrow[ul]
\end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd}[column sep=tiny]
 & 0 & & 0 & & 0 & & 0 & \\
0 \arrow[ur, dash] & & 0 \arrow[ul, dash] \arrow[ur, dash] & & \mathbb{F} \arrow[ul, dash] \arrow[ur, dash] & & 0 \arrow[ul, dash] \arrow[ur, dash] & & 0 \arrow[ul, dash]
\end{tikzcd}
\oplus
\begin{tikzcd}[column sep=tiny]
 & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & & \mathbb{F} & \\
0 \arrow[ur, dash] & & \mathbb{F} \arrow[ul, dash] \arrow[ur, dash] & & \mathbb{F} \arrow[ul, dash] \arrow[ur, dash] & & \mathbb{F} \arrow[ul, dash] \arrow[ur, dash] & & 0 \arrow[ul, dash]
\end{tikzcd}$$

$$\Rightarrow \quad \mathrm{dgm}(H_0 \mathcal{L}^{\leftrightarrow}(\gamma)) = \{ (b,c), [a,d] \}.$$

This is more sophisticated information!

$$\begin{tikzcd}
\mathbb{F} \arrow[d, "(1)"'] & \mathbb{F}^2 \arrow[l, "(1\ 1)"'] \arrow[r, "(1\ 1)"] \arrow[d, "P"] & \mathbb{F} \arrow[d, "1"] \\
\mathbb{F} & \mathbb{F}^2 \arrow[l, "(1\ 0)"] \arrow[r, "(1\ 0)"'] & \mathbb{F}
\end{tikzcd}$$

$\tau(1\ 1) = (1\ 0)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Plan We will follow 3 steps in order.

1. Understanding the statement
   (i) (co) limits.
   (ii) Kan extensions
   today → (iii) Interleaving distance between 2D-modules & some implications.
   (iv) Bottleneck distance between block barcodes.

2. Applications (Proof for II)

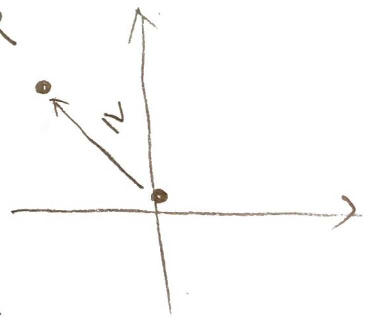
3. Proof — only special case (this is technical enough)

Interleaving of $\mathbb{R}^{op}\times\mathbb{R}$-indexed modules.

Def $\mathbb{R}^{op}\times\mathbb{R}$ is a poset where

i) $a\le a$ $\forall a\in A$. (reflexivity)
ii) $a\le b$ & $b\le a \Rightarrow a=b$ (anti-symmetry)
iii) $a\le b$ & $b\le c \Rightarrow a\le c$ (transitivity)

$$(x,y)\le(x',y') \text{ iff } x\ge x' \ \& \ y\le y' \text{ in } \mathbb{R}$$



Def For any poset $\mathbb{P}$,

$\mathbb{P}$-indexed module refers to a functor $(\mathbb{P},\le)\to \mathrm{Vec}$.

Def (ε-shifting). Given a $\mathbb{R}^{op}\times\mathbb{R}$-module M and $\varepsilon\ge 0$,

Define $M(\varepsilon):\mathbb{R}^{op}\times\mathbb{R}\to\mathrm{Vec}$ by $(M(\varepsilon))_a = M_{a+\vec{\varepsilon}}$.

where $\vec{\varepsilon}:=(-\varepsilon,\varepsilon)$. Also,

$$(M(\varepsilon))(a\le b) = M(a+\vec{\varepsilon}\le b+\vec{\varepsilon}).$$

Def (Morphism between $\mathbb{P}$-indexed modules).

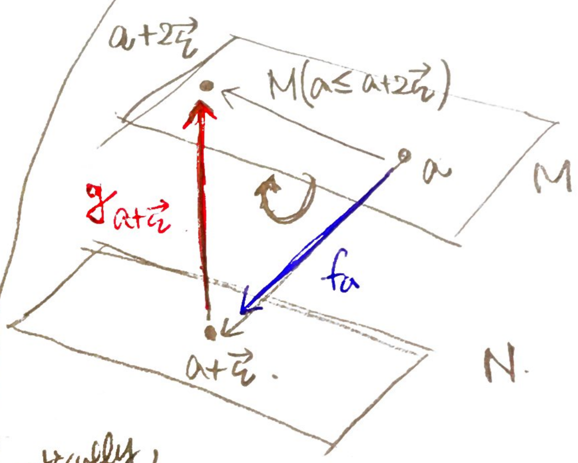
Given $\mathbb{P}$-indexed modules M, N, a morphism $f:M\to N$ is a collection $\{f_a: M_a\to N_a\}_{a\in\mathbb{P}}$ of linear maps such that

$$\begin{tikzcd} M_a \arrow[r, "M(a\le b)"] \arrow[d, "f_a"'] \arrow[dr, phantom, "\circlearrowleft"] & M_b \arrow[d, "f_b"] \\ N_a \arrow[r, "N(a\le b)"'] & N_b \end{tikzcd}$$

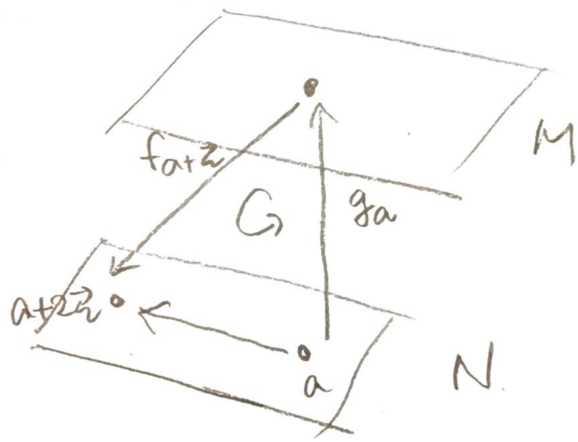
this diagram commutes for all $a \le b$ in $\mathbb{P}$.

<u>Def</u> (Interleaving distance between $\mathbb{R}^{op}\times\mathbb{R}$-indexed modules).

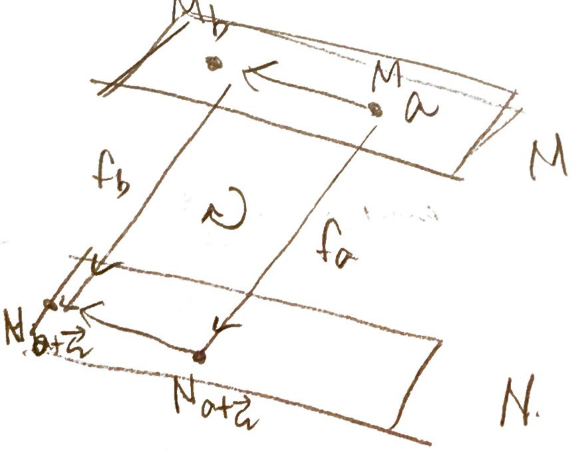
Let M, N be $\mathbb{R}^{op}\times\mathbb{R}$ indexed modules. They are called <u>ε-interleaved</u> if $\exists\ f: M \to N(\varepsilon)$, $\exists\ g: N \to M(\varepsilon)$, for all $a \in \mathbb{R}^{op}\times\mathbb{R}$,



and



Diagrammatically,



<u>We call f, g ε-interleaving morphisms</u>

Define: $d_I(M,N) := \inf\{\varepsilon \ge 0 : M, N \text{ are } \varepsilon\text{-interleaved}\}$.

<u>d_I is an extended pseudometric.</u>