Jan 29th

<u>Recall</u> We defined : for $M, N : \mathbb{R}^{op} \times \mathbb{R} \to \mathrm{Vec}$

$$d_I(M,N) := \inf\{\epsilon \geq 0 : M, N \text{ are } \epsilon\text{-interleaved}\}.$$

An extended pseudo metric.

<u>Remark</u> (i) M, N are ϵ-interleaved, then for all $\epsilon' > \epsilon$, M, N are ϵ'-interleaved.

(ii) Let $M := I^{[0,1] \times [0,1]}$, $N : I^{[0,1) \times [0,1)}$

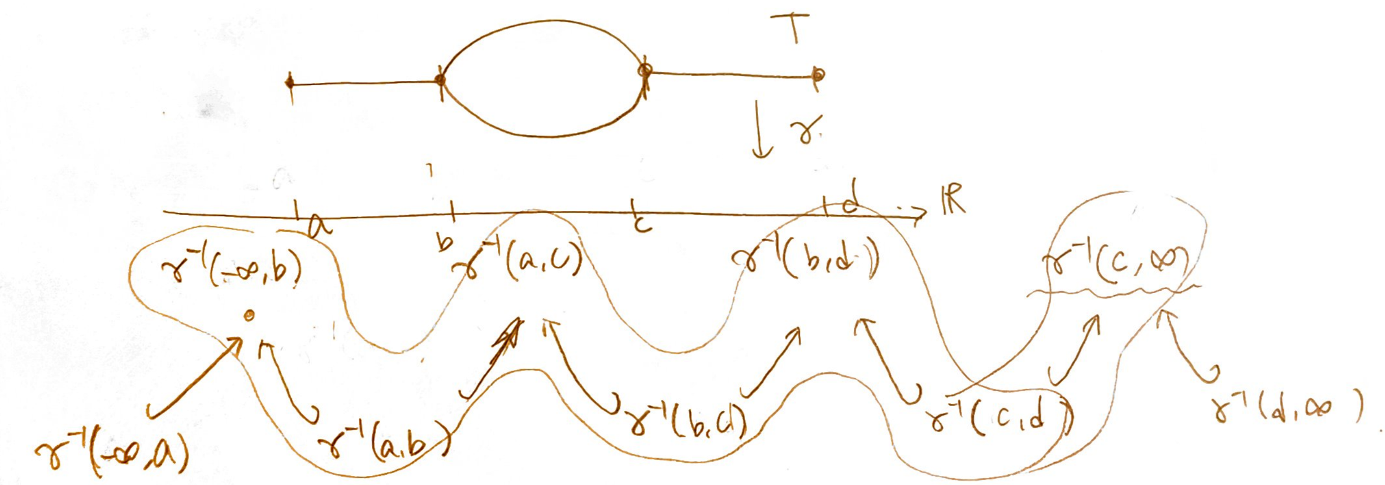
Then M, N are ϵ-interleaved for all $\epsilon > 0$, and hence

$d_I(M,N) = 0$ but $M \neq N$.

(iii) let $M = I^{(-\infty,\infty) \times (-\infty,\infty)}$ and $N = 0$.

Then, there is no interleaving pair, implying $d_I(M,N) = \infty$.

<u>Recall</u> (Zigzag persistent homology)



Recall that $\mathrm{dgm}(H_0(\overleftrightarrow{\mathcal{L}}(\gamma))) = \{[a,d], (b,c)\}$

This makes sense because $\gamma^{-1}(-\infty,b)$ is homotopy equivalent to $\gamma^{-1}(a)$

and $\gamma^{-1}(c,\infty)$ is also h.e to $\gamma^{-1}(d)$.

(See Edelsbrunner and et al, "Zigzag persistent homology and real-valued functions") for details.

Kernels & Cokernels of Morphisms between $\mathbb{P}$-indexed modules

Recall For a linear map $f: V \to W$ between vector spaces V, W,
We have $\ker(f)$ and $\text{coker}(f) = W/\text{Im}(f)$.

The smaller $\ker(f)$ is, The more "faithful" the action of f is.
The smaller $\text{coker}(f)$ is, the more "full" the image of f in W.

Specifically, $\ker(f)=0$ & $\text{coker}(f)=0$
$\Longleftrightarrow V \overset{f}{\cong} W$.

We will define the kernel and cokernel of a morphism between $\mathbb{P}$-indexed modules. These notions again will tell us the "quality" of the morphism as a dissimilarity measure.

Def Let $M, N : \mathbb{P} \to \text{Vec}$. Let $f: M \to N$ be a morphism.
We define $\ker(f)$, $\text{coker}(f) : \mathbb{P} \to \text{Vec}$.

① $\ker(f)$; $\forall a \leq b$ in $\mathbb{P}$,

$$\ker(f)_a := \ker(f_a) \leq M_a.$$

$\ker(f)_a \to \ker(f)_b$ is defined as a restriction $M(a\leq b)|_{\ker(f_a)}$. (*)

②. $\text{coker}(f)$; $\forall a \leq b$ in $\mathbb{P}$

$$\text{coker}(f)_a := \text{coker}(f_a) = N_a / \text{Im}(f_a).$$

$\text{coker}(f)_a \longrightarrow \text{coker}(f)_b$ is defined by $[v] \longmapsto [N(a\leq b)(v)]$. (**)

In order to show these $\mathbb{P}$-indexed modules are well-defined, need to check the following:

① $\forall a \le b$, $M(a\le b)[\ker(f_a)] \le \ker(f_b)$

② $\forall a \le b$, $N(a\le b)[\operatorname{Im}(f_a)] \le \operatorname{Im}(f_b)$.

Use the fact that f is a natural transformation.

<u>Remark/example.</u>

(1) $f: M \to N$ is an isomorphism $\Longleftrightarrow$ $\ker(f)=0$ & $\operatorname{coker}(f)=0$.

(2). For the zero morphism $0: M \to N$, $\ker(f)=M$, $\operatorname{coker}(f)=N$.
(It doesn't tell us anything about structural similarity between M & N)

(3). Only morphism from $I^{\mathbb{R}}$ to $\bigoplus_{r\in\mathbb{R}} \mathbb{R}_r$ is the zero morphism.

<u>Suggestive moral:</u> Existence of "good" morphisms between $\mathbb{P}$-indexed modules M, N would mean structural similarity between M & N.

<u>Def</u> A $\mathbb{R}$-indexed module or $(\mathbb{R}^{op}\times\mathbb{R})$-indexed module M is called ε-trivial if $\forall a\in\mathbb{R}$, $M(a\le a+\varepsilon)$ (or $M(a\le a+\vec{\varepsilon})$) is zero map.

<u>Remark</u> ($M: \mathbb{R}\to \mathrm{Vec}$ is ε-trivial) $\Longleftrightarrow$ (Maximal length of intervals in $\mathrm{dgm}(M)$ $\le \varepsilon$)

$\Longleftrightarrow$ $\left(d_I(M,0) = d_B(\mathrm{dgm}(M), \mathrm{dgm}(0)) \le \frac{\varepsilon}{2}\right)$

, (($\mathbb{R}^{op} \times \mathbb{R}$)-indexed case) (i) Let $f: M \to N(\epsilon)$ be an ϵ-interleaving morphism.

Then $\ker(f)$, $\mathrm{coker}(f)$ are 2ϵ-trivial.

(ii) For $f: M \to N(\epsilon)$, if $\ker(f)$, $\mathrm{coker}(f)$ are 2ϵ-trivial, then

f is a "2ϵ"-interleaving morphism.

(Co)limits of Diagrams

Functors

Motivating example: Given a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Let $\alpha = \inf(a_n)_{n\in\mathbb{N}}$

$\beta = \sup(a_n)_{n\in\mathbb{N}}$. These values somehow "summarize" the sequence $(a_n)_{n\in\mathbb{N}}$.

Considering $n \mapsto a_n$ as a functor $F : \mathbb{N}$ (discrete) $\longrightarrow (\mathbb{R}, \leq)$,

$\alpha = \varprojlim F$ is the limit (or the left root) of F

& $\beta = \varinjlim F$ is the colimit (or the right root) of F.

Universal property of α & β (this justifies why α, β represents/summarize (a_n)).

① $\alpha \leq a_n$ for all $n \in \mathbb{N}$ & $\forall r \in \mathbb{R}\,[r \leq a_n, \forall n\in\mathbb{N} \Rightarrow r \leq \alpha]$

② $\beta \geq a_n$, " & $\forall r \in \mathbb{R}\,[r \geq a_n, \forall n\in\mathbb{N} \Rightarrow r \geq \beta]$

Remark $\left.\begin{array}{l} \text{Inf(empty set)} = +\infty \\ \text{Sup(empty set)} = -\infty \end{array}\right)$ by mathematical logics.

We will generalize the notion of $\varprojlim F$ and $\varinjlim F$ for arbitrary functors F.

Def (Initial & Terminal objects).

An object x in C is initial if $\forall y \in Ob(C)$, $\exists!\ y \to x$.

" terminal if " $\exists!\ x \to y$.

ex). the zero space in Vec is both initial & terminal.

<u>Def</u> (Diagram) Suppose I is a small category (i.e. $ob(I)$ is a set) and C is an arbitrary category. <u>A diagram</u> is simply a functor $F: I \to C$.

Def (cocone). Let $F: I \to \mathcal{C}$ be a diagram. A <u>cocone on F</u> is an object $C \in ob(\mathcal{C})$ together with a collection of morphisms $\phi_x : F(x) \to C$, for each object $x \in ob(I)$, such that for each morphism $g: x \to y$ in I,

$$\begin{tikzcd} & C & \\ F(x) \arrow[ur, "\phi_x"] \arrow[rr, "F(g)"'] & \circlearrowleft & F(y) \arrow[ul, "\phi_y"'] \end{tikzcd}.$$

$(C, \{\phi_x : x \in I\})$

a cocone on F.

e.g. Consider the following diagram in <u>Vec</u>:

$$F: \begin{tikzcd} & \mathbb{R}_1 \arrow[dl, "1"'] \\ \mathbb{R}_2 \arrow[r, "1"'] & \mathbb{R}_3 \arrow[u, "1"'] \end{tikzcd}$$

(i.e. a functor from

$$\begin{tikzcd} & 1 \arrow[dl] \\ 2 \arrow[r] & 3 \arrow[u] \end{tikzcd}$$

to <u>Vec</u>.)

Then,

$$\begin{tikzcd} & \mathbb{R} & \\ \mathbb{R}_2 \arrow[ur, "1_2"] \arrow[dr, "1"'] & & \mathbb{R}_1 \arrow[ul, "1_1"'] \arrow[ll, "1_3"'] \\ & \mathbb{R}_3 \arrow[uu, "1"] \arrow[ur, "1"'] & \end{tikzcd}$$

commutes and hence,

$(\mathbb{R}, \{1_i : \mathbb{R}_i \to \mathbb{R}\}_{i=1}^{3})$ is a cone on F.

Def (The category $Cocone(F)$ of cones of F)

- Objects: cocones $(C, \{\phi_x : F(x) \to C\}_{x \in I})$
- Arrows: An arrow $(C, \{\phi_x : F(x) \to C\}_{x \in I}) \longrightarrow (C', \{\phi'_x : F(x) \to C'\}_{x \in I})$ consists of an arrow $u : C \to C'$ in $\mathcal{C}$ such that

$u \circ \phi_x = \phi'_x$ for all $x \in ob(I)$.