Feb 12th.

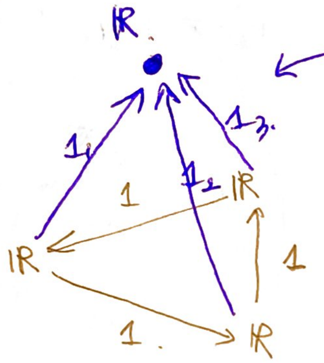
Goal Turn Zigzag modules into $\mathbb{R}^{op}\times\mathbb{R}$-modules.

Recall

Def We call a functor $F: I \to \mathcal{C}$ a diagram.

I: indexing category.

An example of cocone:



$(\mathbb{R}, \{1_1, 1_2, 1_3\})$ a cocone on F.

← a diagram F: $\cdot \swarrow \uparrow \cdot \to \cdot$ $\longrightarrow$ Vec

Def (cocone).

Given a functor F, a cocone of F is $(C, \{\phi_x : F(x) \to C\}_{x\in I})$

satisfying

$$\begin{tikzcd} & C & \\ F(x) \arrow[ur, "\phi_x"] \arrow[rr, "F(g)"'] \arrow[rr, phantom, shift left=4, "\circlearrowleft"] & & F(y) \arrow[ul, "\phi_y"'] \end{tikzcd}$$

for each $g: x \to y$.

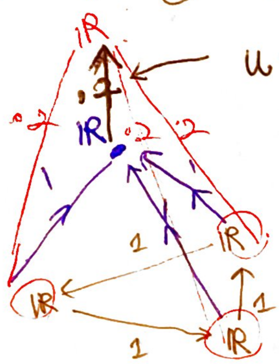
Def (Category of cocones on F) Fix a functor $F: I \to \mathcal{C}$.

The category $\mathrm{Cocone}(F)$ of cocones of F consists of:

① Objects: Cocones $(C, \{\phi_x : F(x) \to C\}_{x\in I})$ on F

② Arrows: an arrow $(C, \{\phi_x\}) \to (C', \{\phi'_x\})$ is an arrow $u: C \to C'$ in $\mathcal{C}$ s.t. $\phi'_x = u \circ \phi_x$ for all $x \in ob(I)$.

e.g.



$u: \mathbb{R} \to \mathbb{R}$.

Def (Colimit) The colimit of a diagram $F : I \to \mathcal{C}$ is the initial object in cocone(F). In other words, to say that $(\varinjlim F, \{\phi_x : x \in I\})$ is the colimit of F means that for any cocone $(c, \{\phi'_x : x \in I\})$, $\exists!\, u : \varinjlim F \to c$ s.t. $\forall g : x \to y$,

$$\begin{tikzcd}
 & c & \\
 & \varinjlim F \arrow[u, dashed, "\exists!\, u"] & \\
F(x) \arrow[ur, "\phi_x"] \arrow[uur, bend left, "\phi'_x"] \arrow[rr, "F(g)"'] & & F(y) \arrow[ul, "\phi_y"'] \arrow[uul, bend right, "\phi'_y"']
\end{tikzcd}$$

Remark Often times, one refers to only the object $\varinjlim F$ by a colimit of F.

e.g. $F : \mathbb{N}$ (discrete) $\to (\mathbb{R}, \le)$. (i.e. a sequence in $\mathbb{R}$)

- Any upper bound is a cocone of F (if there is any)
- The l.u.b is the colimit of F.

e.g. (Coproduct) The colimit of any functor $F : \underset{x}{\bullet} \quad \underset{y}{\bullet} \to \mathcal{C}$ is called coproduct. (More generally, when the domain is discrete).

$\mathcal{C} = \textbf{Sets}$; disjoint unions.

$\mathcal{C} = P$ (a poset) ; the l.u.b.

$\mathcal{C} = \textbf{Vec}$; direct sum.

$(u, v) \longmapsto (f(u), g(v))$

$$\begin{tikzcd}
 & W & \\
 & U \oplus V \arrow[u, "(f,g)"] & \\
U \arrow[ur, "i_1"] \arrow[uur, bend left, "f"] & & V \arrow[ul, "i_2"'] \arrow[uul, bend right, "g"']
\end{tikzcd}$$

e.g. (Pushouts). In **set**,

$$\varinjlim \left(\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd} \right) = B \amalg C / \sim$$

$b \sim c$ iff $\exists\, a \in A$, $b = f(a)$ & $c = g(a)$

In vec,

$$\varinjlim \left(\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle g}\downarrow & & \\ W & & \end{array} \right) = V \oplus W / \mathrm{Im}(f, -g)$$

where

$$(f,-g) : U \to V \oplus W$$
$$u \mapsto (f(u), -g(u))$$

e.g. In any category $\mathcal{C}$,

$$\varinjlim \left(\begin{array}{ccc} & & A \\ & & \downarrow f \\ B & \xrightarrow[g]{} & C \end{array} \right) = C.$$

If there is a "sink" in a diagram, then that sink is the colimit of that diagram

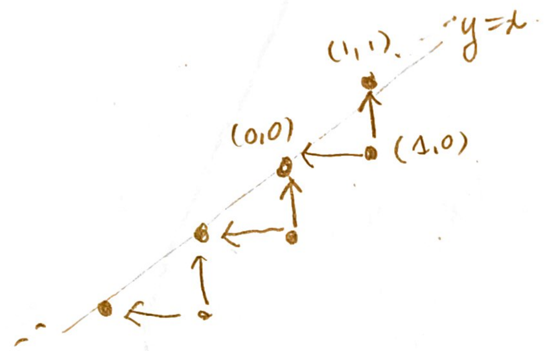
e.g. (cokernel)

$\varinjlim \left(V \underset{0}{\overset{f}{\rightrightarrows}} W \right)$ is $\mathrm{coker}(f) = W / \mathrm{Im}(f)$.

(c.f. $\varprojlim \left(V \underset{0}{\overset{f}{\rightrightarrows}} W \right)$ is $\ker(f)$).

Transforming $\mathbb{ZZ}$-modules to $(\mathbb{R}^{op}\times\mathbb{R})$-indexed modules

Def (Poset $\mathbb{ZZ}$) Define a poset $\mathbb{ZZ}$ (as a subposet of $\mathbb{R}^{op}\times\mathbb{R}$) as follows: $\mathbb{ZZ} = \{(i,j) : i\in\mathbb{Z},\ j=i \text{ or } i-1\}$.

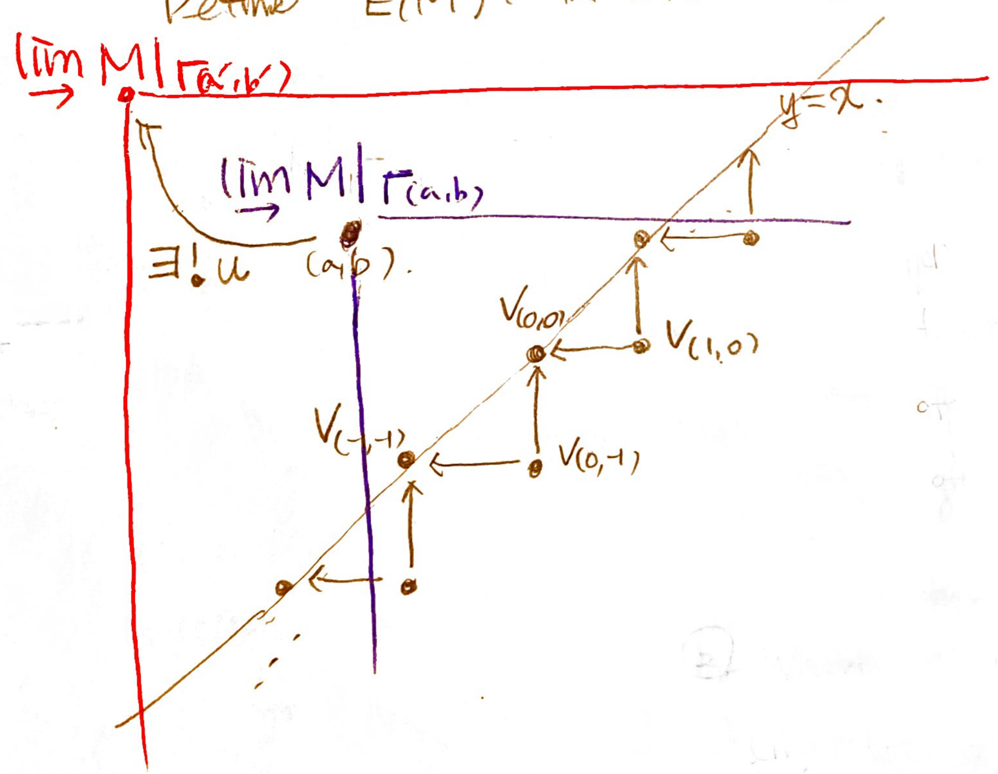


We will call $\mathbb{ZZ}$-indexed modules zigzag modules.

Def For any poset $\mathbb{P}$, let $\mathrm{Vec}^{\mathbb{P}}$ be the category of $\mathbb{P}$-indexed modules with natural transformations.

Def (Embedding functor $E : \mathrm{Vec}^{\mathbb{ZZ}} \to \mathrm{Vec}^{\mathbb{R}^{op}\times\mathbb{R}}$) Let $M : \mathbb{ZZ} \to \mathrm{Vec}$.

Define $E(M) : \mathbb{R}^{op}\times\mathbb{R} \to \mathrm{Vec}$ as follows:



More explanations about remarks above.

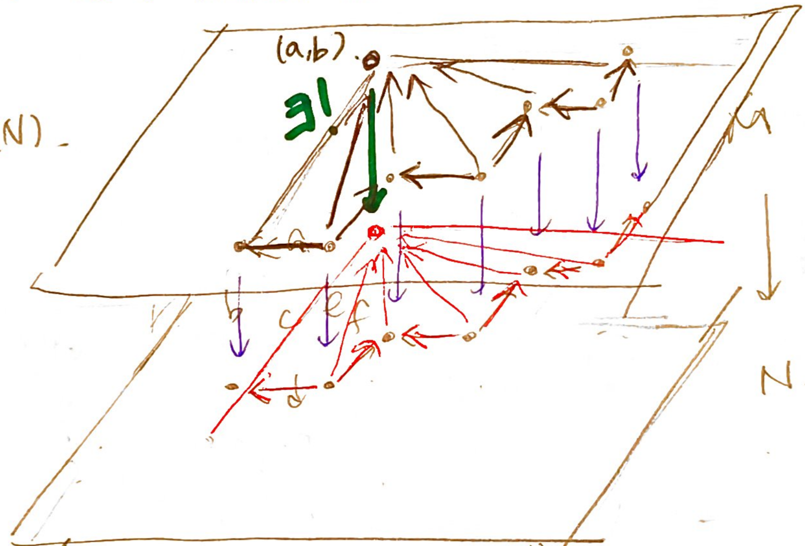
① For each $(a,b) \in \mathbb{R}^{op}\times\mathbb{R}$, $E(M)(a,b)$ is uniquely defined by $\varinjlim M|_{\Gamma(a,b)}$ (← this exists since Vec is cocomplete). In the picture, $\varinjlim M|_{\Gamma(a',b')}$ is also a cone on $M|_{\Gamma(a,b)}$ and thus there exists a unique arrow $\varinjlim M|_{\Gamma(a,b)} \xrightarrow{u} \varinjlim M|_{\Gamma(a',b')}$ by the UP of the colimit.

$\{(k,l)\in \mathbb{ZZ} : i(k,l)\le (a,b)\}$

② the Colimit of empty diagram in $\mathcal{C}$ is the initial object of $\mathcal{C}$.

③ (Functoriality) Let $M, N : \mathbb{ZZ} \to \mathbf{Vec}$ and let $f : M \to N$ be a natural transformation.

Let us define $E(f) : E(M) \to E(N)$.



Note that $\left(\varinjlim N|_{\Gamma(a,b)},\ \psi|_{\Gamma(a,b)} \circ f|_{\Gamma(a,b)}\right)$ is a cone on $M|_{\Gamma(a,b)}$,

by the universal property of the colimit $\varinjlim M|_{\Gamma(a,b)}$,

there is a unique arrow $u_{(a,b)} : \varinjlim M|_{\Gamma(a,b)} \longrightarrow \varinjlim N|_{\Gamma(a,b)}$.

Remark (Kan extensions)

This way of extension is called left-Kan extension of M along $i : \mathbb{ZZ} \to \mathbb{R}^{op}\times\mathbb{R}$.

This extension preserves direct sums in $\mathbf{Vec}^{\mathbb{ZZ}}$.

∵ (① $E : \mathbf{Vec}^{\mathbb{ZZ}} \to \mathbf{Vec}^{\mathbb{R}^{op}\times\mathbb{R}}$ is left adjoint to Restriction $\mathbf{Vec}^{\mathbb{R}^{op}\times\mathbb{R}} \to \mathbf{Vec}^{\mathbb{ZZ}}$

② If a functor is a left adjoint, then it preserves colimits.)